Lecture 9

Recall the notions of G_δ vs. F_σ sets (see §1.2 in Folland):

- E is a G_δ set if $E = \bigcap_{n=1}^{\infty} U_n$ where U_n's are open.
- E is a F_σ set if $E = \bigcup_{n=1}^{\infty} F_n$ where F_n's are closed.

These appear naturally if one tries to describe the Borel σ-algebra by construction.

Thm 2 TFAE:

(i) $E \in \mathcal{M}_{\mu_F}$

(ii) $E = V \setminus N$, V is G_δ, $\mu_F(N) = 0$

(iii) $E = W \cup N'$, W is F_σ, $\mu_F(N') = 0$.

Pf. Since μ_F is complete, both (ii) and (iii) $\Rightarrow$ (i).

To prove converse, suppose first $\mu_F(E) < \infty$. By previous thm, $\exists$ open U_n, compact K_n s.t. $K_n \subseteq E \subseteq U_n$ and

$$\mu_F(K_n) + \frac{1}{n} \leq \mu_F(E) \leq \mu_F(U_n) - \frac{1}{n}$$

Taking $V = \bigcap_{n=1}^{\infty} U_n$, $W = \bigcup_{n=1}^{\infty} K_n$, we get G_δ and F_σ s.t. $W \subseteq E \subseteq V$ and $\mu_F(E) = \mu_F(W) = \mu_F(V)$. Now, letting $N = V \setminus E$ and $N' = E \setminus W$ and using completeness of μ_F, we conclude (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i). $\boxtimes$

The Lebesgue measure.

We now consider the special case where $F(x)=x \Rightarrow \mu_F((a,b])=b-a$. This is simply known as the Lebesgue measure (LM). Following Folland, we will denote the LM by m and its σ-algebra by $\mathcal{L}$.

The LM has invariance properties under natural symmetries of $\mathbb{R}$. Let $E\in\mathcal{L}$, and $s,a\in\mathbb{R}$. Then

$$a+E=\{a+x : x\in E\},\quad sE=\{sx : x\in E\}.$$

Prop 1. If $E\in\mathcal{L}$, then $a+E$, $rE\in\mathcal{L}$ and $m(a+E)=m(E)$, $m(rE)=|r|\cdot m(E)$.

Pf. See Folland. $\boxtimes$

Cantor Sets

We construct a set $\mathcal{C} \subseteq [0,1]$ as follows. Let $\mathcal{C}_0 = [0,1]$. $\mathcal{C}_1$ is obtained be removing the "open middle third" of the interval in $\mathcal{C}_0$, i.e.

$$\mathcal{C}_1 = [0,1] \setminus (\tfrac{1}{3}, \tfrac{2}{3}).$$

Assume $\mathcal{C}_k \subseteq [0,1]$ is a disjoint union of 2^k closed intervals of length $\frac{1}{3^k}$. $\mathcal{C}_{k+1}$ is obtained by removing the "open middle thirds" of each interval in $\mathcal{C}_k$. Then $\mathcal{C}_{k+1}$ satisfies the assumption also and we construct by assumption a sequence $\{\mathcal{C}_k\}_{k=0}^{\infty}$ of such sets. Moreover $\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \dots \supseteq \mathcal{C}_k \supseteq \dots$

$$\boxed{\mathcal{C} = \bigcap_{k=1}^{\infty} \mathcal{C}_k \ .} \quad \text{(Cantor set)}.$$

By cont. from above, we have

$$m(\mathcal{C}) = \lim_{k\to\infty} m(\mathcal{C}_k) = \lim_{k\to\infty} \left(\frac{2}{3}\right)^k = 0.$$

One can generalize the construction above by choosing a sequence $\{\alpha_k\}_{k=1}^{\infty}$ such that $\alpha_k \in (0,1)$.

At each step above, remove instead the "open middle $\alpha_k$ith interval". Then each $\mathcal{C}_k^{\alpha}$ consists of 2^k closed intervals of length $\frac{(1-\alpha_1)\cdot \ldots \cdot (1-\alpha_k)}{2^k}$. Thus,

$$m\left(\mathcal{C}^{\alpha} = \bigcap_{k=1}^{\infty} \mathcal{C}_k^{\alpha}\right) = \lim_{k\to\infty} \prod_{j=1}^{k} (1-\alpha_k)$$

$$=: \prod_{k=1}^{\infty} (1-\alpha_k) = e^{\sum_{k=1}^{\infty} \log(1-\alpha_k)},$$

where $e^{\sum \log(1-\alpha_n)} = \begin{cases} 0, & \sum|\log(1-\alpha_n)| = \infty \\ >0, & \sum|\log(1-\alpha_n)| < \infty \end{cases}$

always < 0

Note that for $0 < t << 1$

$$\frac{1}{2}t \leq |\log(1-t)| \leq \frac{3}{2}t.$$

Thus, if $\sum_{n=1}^{\infty} \alpha_n < \infty$, then

$m(C^{\alpha}) > 0$.

For all Cantor sets, it holds that

<u>Prop 3</u>. C is compact and contains no intervals ($\Rightarrow$ nowhere dense and totally discontinuous). No point in C is isolated.